Coupled dust-lattice solitons in monolayer plasma crystals

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Nonlinearly coupled dust-lattice (DL) waves in monolayer plasma crystals are studied theoretically. It is shown that the high-frequency transverse (vertical) oscillations can form localized wave envelopes—solitons coupled with ‘‘slow’’ longitudinal DL perturbations. Using the molecular dynamics simulations, the derived soliton solution is shown to be stable.

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Recently, a series of papers has been published where experiments on various nonlinear phenomena in complex (dusty) plasmas, e.g., Mach cones, solitons, shocks, etc., are reported [1–3]. Complex plasmas allow us to study different physical processes at the kinetic level [4,5]. In particular, when (negatively) charged microparticles are strongly coupled and form ordered structures, ‘‘plasma crystals’’ [6,7], one can investigate the kinetics of nonlinear processes in crystalline lattices. Plasma crystals ‘‘grown’’ under microgravity conditions do not experience ‘‘self-compression’’ caused by the particle weight. Structures of such crystals formed in isotropic stress-free conditions are very similar to those observed in solid states, and therefore plasma crystals can be considered as a ‘‘test sample’’ for the kinetic study of, e.g., phase transitions in Yukawa systems [8]. At the same time, in ground-based experiments particles levitate in regions with strongly inhomogeneous vertical electric fields (e.g., in sheaths of rf electrodes), where the electric force balances gravity. Under these conditions, one can easily form crystalline monolayers which are very convenient for wave analysis [9,10].

In this paper we investigate theoretically the nonlinearly coupled dust-lattice (DL) waves in monolayer plasma crystals. We show that the transverse (vertically polarized) waves can form localized wave envelopes—coupled DL solitons. Due to slow longitudinal DL perturbations induced by the vertical oscillations, the particle density is increased in the soliton. The derived soliton solution is compared with results of molecular dynamics simulations and is shown to be stable.

The motion of the charged particles is determined by the electrostatic interaction, which is the sum of the interparticle coupling and the interaction with the external confinement field. For particles suspended in rf sheaths, a screened potential of the Yukawa type was shown to be a reasonable approximation for the interaction in a horizontal direction [11]. Also, the experimentally observed mean interparticle distance Δ_0 is typically larger than the screening length λ. Therefore, it is usually sufficient to take into account only the ‘‘nearest neighbor’’ interaction.

For the analysis of waves in crystalline monolayers, we use the so-called ‘‘particle string’’ model, which allows two-dimensional motion, in the longitudinal (horizontal, along the string axis) and transverse (vertical) directions [12,13]. We assume no longitudinal confinement and a harmonic potential well in the vertical direction, with eigenfrequency Ω. The nearest neighbor approximation along with the particle string model allows us to simplify the resulting equations substantially and, thus, to make the physics more ‘‘transparent.’’ The total interaction energy for a particle is $U_\Sigma = U_+ + U_- + \frac{1}{2}M\Omega^2 z^2$, where $U_\pm = (Q^2/\Delta_\pm)e^{-\Delta_\pm/\lambda}$ is the coupling of the ‘‘central’’ particle with the ‘‘right’’ and the ‘‘left’’ neighbor, respectively, Q is the particle charge, and M is the mass. Introducing the particle displacement in the horizontal and vertical directions, $\mathbf{r}=\{x,z\}$ (similarly, $\mathbf{r}_\pm=\{x_\pm,z_\pm\}$ for the right and the left neighbor, respectively), we get the interparticle distance $\Delta_\pm=\sqrt{(\Delta_0+\delta x_\pm)^2+\delta z_\pm^2}$, which depends on the *relative displacement* with respect to the right, $\delta\mathbf{r}_+=\mathbf{r}_+-\mathbf{r}$, and to the left, $\delta\mathbf{r}_-=\mathbf{r}-\mathbf{r}_-$, neighbor.

The equation of the particle motion is $M\ddot{\mathbf{r}}=-\partial U_\Sigma/\partial\mathbf{r}$. Expanding the coupling energy in a series over the relative displacements $\delta x_\pm$ and $\delta z_\pm$, we derive the following equations for the horizontal and vertical motion, respectively:

$$\begin{aligned}\ddot{x} = & -2\kappa^{-3}(1+\kappa+\tfrac{1}{2}\kappa^2)e^{-\kappa}(\delta x_- \delta x_+)\\ & +3\kappa^{-4}(1+\kappa+\tfrac{1}{2}\kappa^2+\tfrac{1}{6}\kappa^3)e^{-\kappa}(\delta x_-^2-\delta x_+^2)\\ & -\tfrac{3}{2}\kappa^{-4}(1+\kappa+\tfrac{1}{3}\kappa^2)e^{-\kappa}(\delta z_-^2-\delta z_+^2)\\ & +O(\delta x_\pm^3,\delta x_\pm \delta z_\pm^2,\delta z_\pm^4), \end{aligned} \quad (1)$$

$$\begin{aligned}\ddot{z}+\Omega^2 z = & \,\kappa^{-3}(1+\kappa)e^{-\kappa}(\delta z_- \delta z_+)\\ & -\tfrac{3}{2}\kappa^{-5}(1+\kappa+\tfrac{1}{3}\kappa^2)e^{-\kappa}(\delta z_-^3-\delta z_+^3)\\ & -3\kappa^{-4}(1+\kappa+\tfrac{1}{3}\kappa^2)e^{-\kappa}(\delta x_-\delta z_- - \delta x_+\delta z_+)\\ & +O(\delta x_\pm^2\delta z_\pm,\delta z_\pm^5), \end{aligned} \quad (2)$$

where $\kappa=\Delta_0/\lambda \gtrsim 1$ is the lattice parameter. Here and below, the displacements are normalized by the screening length, $\mathbf{r}/\lambda\to\mathbf{r}$, time as well as all frequencies are normalized to $\Omega_{\rm DL}t\to t$ and $\Omega/\Omega_{\rm DL}\to\Omega$, where $\Omega_{\rm DL}^2=Q^2/M\lambda^3$ is the DL frequency scale. In the linear regime Eqs. (1) and (2) yield the well-known dispersion relations for the longitudinal ($\parallel$, horizontal) and transverse ($\perp$, vertical) modes [13,14],

$$\omega_\parallel^2(K)=4\Omega_\parallel^2\sin^2(\tfrac{1}{2}\kappa K),$$

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$$\omega_\perp^2(K)=\Omega^2-4\Omega_\perp^2\sin^2(\tfrac{1}{2}\kappa K), \tag{3}$$

with $\Omega_\parallel^2=2\kappa^{-3}(1+\kappa+\frac{1}{2}\kappa^2)e^{-\kappa}$ and $\Omega_\perp^2=\kappa^{-3}(1+\kappa)e^{-\kappa}$ the corresponding frequency scales. The wave vector is normalized by the screening length $\lambda K\to K$, so that the K range in the first Brillouin zone is [15] $0\leqslant\kappa K\leqslant\pi$.

The transverse wave mode $\omega_\perp(K)$ has an optical branch and the longitudinal mode $\omega_\parallel(K)$ is described by an acoustic branch. Usually, the resonance frequency of the vertical oscillations exceeds substantially the frequency scales of both modes [2,13,16], $\Omega\gg\Omega_\parallel\sim\Omega_\perp$. Therefore in a nonlinear regime, when the modes are strongly coupled, the vertical oscillations can be treated as a source of a "high-frequency pressure" for the relatively slow longitudinal motion. Variation of the particle density (along the string) which is induced by these oscillations is positive, i.e., the "pressure" is negative: The density grows because the relative vertical displacements increase and, hence, the horizontal repulsion between neighbors becomes weaker.

Let us first consider how the transverse oscillations would evolve in a "quasilinear" regime. The analysis can be conveniently done in terms of the Hamiltonian formalism [17], considering vertical oscillations as quasiparticles—*oscillatons*—with energy $\omega_\perp(K)$. Equations of motion for the oscillaton in the Hamiltonian form yield $\dot{K}=-\partial\omega_\perp/\partial X \simeq 4(\Omega_\perp/\Omega)(\partial\Omega_\perp/\partial X)\sin^2(\frac{1}{2}\kappa K)$ and $\dot{X}=\partial\omega_\perp/\partial K\simeq -(\Omega_\perp^2/\Omega)\sin(\kappa K)$, where X is the horizontal coordinate. The frequency scale $\Omega_\perp$ is a steep (exponential) function of the particle density n, with $\partial\Omega_\perp/\partial n>0$. Hence, the force acting on oscillatons (which is proportional to $\dot{K}$) accelerates them towards $\partial n/\partial X>0$. This means that the region of higher density is a "potential well" for the quasiparticles. Accumulation of oscillatons in the region with higher density causes further density increase, and therefore a modulational instability of coupled DL waves might be possible. The instability should be more efficient for shorter wavelengths, since the force on oscillatons is proportional to $\sin^2(\frac{1}{2}\kappa K)$. One can easily find an analogy with other types of modulational instability, e.g., of the Langmuir waves (plasmons) [17–19]. The sign of dispersion is positive for plasmons, $\partial\omega_{\rm L}/\partial K>0$ [since $\omega_{\rm L}^2(K)=\omega_{\rm p}^2+3K^2v_{T_e}^2$, where $\omega_{\rm p}(n)$ is the electron plasma frequency and v_{T_e} is the thermal velocity], but is negative for the DL oscillatons. At the same time, oscillatons cause an increase of the density, whereas the high-frequency pressure of plasmons creates cavities. Also, the force on oscillatons is due to the gradient of the dispersion (K dependent) term in Eq. (3), but for plasmons it is determined by the gradient of $\omega_{\rm p}(n)$. Therefore, the instability develops at shorter wavelengths for oscillatons and at longer wavelengths for plasmons, but its physical mechanism remains essentially the same.

Now we derive equations for the coupled DL modes. We consider the case when the spectrum of the vertical oscillations is determined by a certain primary frequency ω, with some wave number K. Equations (1) and (2) show that the secondary harmonics generated in horizontal motion can have only even indices, and for the vertical oscillations only odd indices. Therefore, one can present the perturbations in the form $x=u(X,t)+B(X,t)e^{2i(\omega t-KX)}+\cdots$, and $z=A(X,t)e^{i(\omega t-KX)}+C(X,t)e^{3i(\omega t-KX)}+\cdots$, where u, A, B, etc., are "smooth" functions of X and t (i.e., $|\partial/\partial t|\ll\omega$ and $|\partial/\partial X|\ll K$). The first longitudinal term u represents the average particle displacement in the string ($-\partial u/\partial X$ is the average density variation), the first transverse term is determined by the amplitude A of vertical oscillations at the primary frequency. In order to describe the average density variation in the string, we can retain in Eq. (1) only linear x terms and the first coupling term $\propto\delta z_\pm^2$. The latter represents the high-frequency pressure of oscillatons and is mostly determined by the primary frequency term. This yields the following equation for the average displacement:

$$\frac{\partial^2u}{\partial t^2}-C_\parallel^2\frac{\partial^2u}{\partial X^2}=C_\parallel^2\mu_\parallel\frac{\partial|A|^2}{\partial X}, \tag{4}$$

where $C_\parallel^2\equiv(\omega_\parallel/K)^2|_{K\to0}=\kappa^2\Omega_\parallel^2$ is the (squared) long-wavelength longitudinal phase velocity and $\mu_\parallel=6\kappa^{-2}[(1+\kappa+\frac{1}{3}\kappa^2)/(1+\kappa+\frac{1}{2}\kappa^2)]\sin^2(\frac{1}{2}\kappa K)$ is the longitudinal coupling coefficient. For the oscillations at the primary frequency, we substitute $z=A(X,t)e^{i(\omega t-KX)}+\text{c.c.}$ in Eq. (2) and get the equation for the complex amplitude:

$$(\omega^2-\omega_\perp^2)A=\frac{\partial^2A}{\partial t^2}+\Omega_\perp^2\left(\ell_{\rm disp}^2\frac{\partial^2A}{\partial X^2}+\Lambda|A|^2A+\mu_\perp A\frac{\partial u}{\partial X}\right)+2i\left(\omega\frac{\partial A}{\partial t}-\Omega_\perp^2\kappa\sin(\kappa K)\frac{\partial A}{\partial X}\right), \tag{5}$$

where $\ell_{\rm disp}^2=\kappa^2\cos(\kappa K)$ is the (squared) dispersion coefficient ("dispersion length"), $\Lambda=72\kappa^{-2}[(1+\kappa+\frac{1}{3}\kappa^2)/(1+\kappa)]\sin^4(\frac{1}{2}\kappa K)$ is the nonlinear coefficient, and $\mu_\perp=12[(1+\kappa+\frac{1}{3}\kappa^2)/(1+\kappa)]\sin^2(\frac{1}{2}\kappa K)$ is the transverse coupling coefficient. In Eq. (5) we omitted the higher-order derivatives and nonlinear terms. Again, one can easily see a similarity between Eqs. (4), (5), and the equations for the Langmuir modulational instability [18,20]. Equation (4) is equivalent to the ion acoustic wave equation coupled to the electric field of the Langmuir wave via the high-frequency pressure (with $-\partial u/\partial X$ as the plasma density perturbation and A as the electric field of the Langmuir wave). Equation (5) is similar to the equation for the Langmuir wave, with the coupling $\propto A\partial u/\partial X$ due to the acoustic density modulation. The difference is the nonlinear term $\propto|A|^2A$.

One can easily see that the nonlinear and coupling terms in Eq. (5) are of the same order. Eq. (4) yields $-\partial u/\partial X\propto|A|^2$. Therefore, the modulational instability can be suppressed due to nonlinearity. Furthermore, a balance between the dispersion term $\propto\partial^2A/\partial X^2$ and the resulting nonlinear term can provide spatial localization of vertical oscillations, i.e., formation of a solitary wave. Possible soliton solution of Eqs. (4) and (5) should depend on the self-similar variable $\xi=X-Ut$, so that $u=u(\xi)$ and $A=A(\xi)$. The complex amplitude is $A=|A|e^{i\Psi}$, where the "phase" Ψ is generally a function of X and t. For the soliton solution we should consider $\Psi=\text{const}$. The transverse oscillations are already taken into account via the factor $e^{i(\omega t-KX)}$, and the additional spatial (temporal) variation of the phase would just "renormalize" values of ω and k. Hence, one can set $A\equiv|A|$. The imaginary terms in Eq. (5) (those in parentheses with prefactor $2i$) readily give an equation of characteristics for $A(X,t)=\text{const}$, i.e., determine the soliton velocity. We get $U=-\kappa(\Omega_\perp^2/\omega)\sin(\kappa K)\equiv\partial\omega_\perp/\partial K$, which means that the soliton velocity is equal to the group velocity of the transverse mode. We see that U is inversely proportional to the primary frequency ω. Recall that we consider the case $\omega\simeq\Omega\gg1$, and therefore U is much smaller than the longitudinal acoustic velocity $C_\parallel$. Then, neglecting terms $O(\Omega^{-2})$ in Eqs. (4) and (5) we finally derive the equations for the oscillation amplitude A and the density variation $-u'$ in the soliton,

$$\ell_{\mathrm{disp}}^2 A''=\frac{\delta\omega^2}{\Omega_\perp^2}A-(\Lambda-\mu_\perp\mu_\parallel)A^3,\qquad -u'=\mu_\parallel A^2,\quad (6)$$

where $\delta\omega^2\equiv\omega^2-\omega_\perp^2$ is a deviation of the (squared) frequency from the linear dispersion. Weak nonlinearity which is assumed in Eq. (5) [and, hence, in Eq. (6)] requires $\delta\omega^2$ to be sufficiently small. The nonlinear coefficient is modified due to the coupling, but remains always negative: $\Lambda-\mu_\perp\mu_\parallel=12\{(1+\kappa+\frac{1}{3}\kappa^2)/[(1+\kappa)(1+\kappa+\frac{1}{2}\kappa^2)]\}\sin^4(\frac{1}{2}\kappa K)>0$. Therefore the soliton solution is only allowed for positive frequency deviation, so that $0<\delta\omega^2/\omega^2\ll1$. (This frequency deviation plays the role of a control parameter for the coupled DL soliton, in analogy to the Mach number excess over unity, M^2-1, which is the control parameter for the longitudinal DL soliton [2].) The dispersion length should be positive as well, i.e., $\cos(\kappa K)>0$, which imposes the upper edge for the wave number $\kappa K<\pi/2$. This is because the dispersion of the transverse mode changes the sign at this point; the branch $\omega_\perp(K)$ becomes concave, so that nonlinearity and dispersion are no longer balancing each other.

The soliton solution of Eq. (6) has the following functional form:

$$A(\xi)=A_0\cosh^{-1}(\xi/L),\qquad -u'=-u_0'\cosh^{-2}(\xi/L),\quad (7)$$

where A_0 and L are the soliton amplitude and width, respectively, and $-u_0'$ is the depth of the density variation. The latter is related to the soliton amplitude via $-u_0'=\mu_\parallel A_0^2$. Substituting these functions in Eq. (6) we obtain $A_0^2=2(\Lambda-\mu_\perp\mu_\parallel)^{-1}(\delta\omega^2/\Omega_\perp^2)$ and $L^2=\ell_{\mathrm{disp}}^2(\Omega_\perp^2/\delta\omega^2)$, or in terms of the lattice parameter:

$$-u_0'=\frac{2e^\kappa}{\kappa\sin^2(\frac{1}{2}\kappa K)}\,\delta\omega^2,$$

$$A_0^2=\frac{\kappa e^\kappa}{3\sin^4(\frac{1}{2}\kappa K)}\left(\frac{1+\kappa+\frac{1}{2}\kappa^2}{1+\kappa+\frac{1}{3}\kappa^2}\right)\delta\omega^2,$$

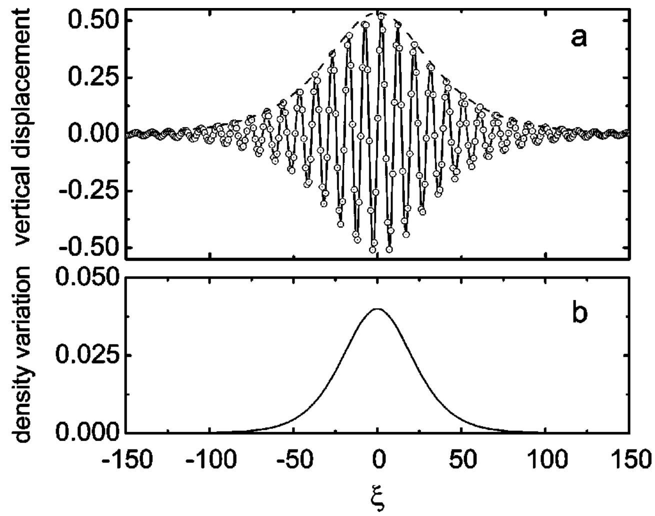

FIG. 1. Molecular dynamics simulations of the coupled DL soliton in a particle string. The number of particles in the simulation is 600. Dimensionless parameters (see text) are vertical resonance frequency $\Omega=40$, deviation of the squared frequency $\delta\omega^2=7\times10^{-4}$, lattice parameter $\kappa=1$, and wave vector $K=0.64$. The shown results are for the time $\Omega t=1000$. (a) Vertical displacements of particles vs the horizontal coordinate ξ, obtained in the simulation (dots connected by solid line) and the theoretical amplitude profile [$A(\xi)$ from Eqs. (7) and (8), dashed line]. Vertical oscillations are at the primary frequency ω. (b) Average variation of the particle density along the string obtained in the simulation. It coincides with the theoretical curve [$-u'(\xi)$ from Eqs. (7) and (8)] with an accuracy $\sim10^{-2}\%$.

$$L^{-2}=\frac{\kappa e^\kappa}{(1+\kappa)\cos(\kappa K)}\,\delta\omega^2.\quad (8)$$

The obtained results formally correspond to the limit $\Omega\to\infty$ (when $U\to0$), but in fact they are valid as soon as $\Omega^2\gg1$. As we mentioned above, this limit physically means that the resonance frequency of vertical oscillations is much higher than the frequency scale of the DL wave, Ω_{DL}. This is true for experiments with crystalline monolayers, where the resonance frequency is usually 15–17 Hz and $\Omega_{\mathrm{DL}}/2\pi$ is about a few hertz [2,10,16]. In this limit we can also neglect higher harmonics in perturbations, which scale as $\propto(m\omega)^{-2}A^m$ (where m is the harmonic index), in particular, the second longitudinal harmonic $Be^{2i(\omega t-KX)}$. Since $B\propto\omega^{-2}A^2$, the terms $AB\propto A^3$ in Eq. (2) formally should affect Eq. (5) for the primary frequency amplitude, but the smallness of ω^{-2} allows us to omit them.

In order to check the stability of the coupled DL soliton, we also performed molecular dynamics simulations in a one-dimensional particle string. For this purpose we solved equations of motion for particles interacting via the Yukawa potential (with the nearest neighbor coupling). No confinement was applied along the string, in the vertical direction the particles were confined in the harmonic potential, and the horizontal transverse motion was restricted. Equations (7) and (8) were chosen as the initial conditions. We solved the equations using the standard Runge-Kutta algorithm with adaptive time step. Figure 1 shows one example with the profile of the transverse amplitude A as well as the induced density variation along the string, $-u'$, obtained in the simulations. One can see that the perturbations in the form of the theoretical solution [Eqs. (7) and (8)] appear to be stable.

In the above derivation we neglected dissipation, in particular, the role of neutral friction on particles. In fact, one can speak about ''weakly dissipative solitons,'' when the dissipation time scale is much longer than the time scale of the soliton formation, which is of the order of $\Omega_{\mathrm{DL}}^{-1}$. In this case, the soliton decays in time, but its form is always ''adjusted'' to the ''true'' soliton solution corresponding to a momentary amplitude. For instance, in recent experiments with longitudinal DL solitons [2], the dissipation time scale due to the Epstein drag is about 0.3–0.5 s, whereas $\Omega_{\mathrm{DL}}^{-1} \sim 0.01$–$0.03$ s, so that the observed decaying perturbations are very well described by the ''nondissipative'' solution.

In conclusion, we showed that the nonlinear longitudinal and transverse oscillations in a monolayer plasma crystal can form a coupled DL soliton, a spatially localized transverse wave envelope with the increased particle density. Numerical simulations suggest that the derived soliton solution is stable.

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